

Oriental

Richard Steiner

For Ross Street on his sixtieth birthday

ABSTRACT. The orientals or oriented simplexes are a family of strict omega-categories constructed by Ross Street. We show that the category of orientals is isomorphic to a subcategory of the category of chain complexes. This leads to a very simple combinatorial description of the morphisms between orientals. We also show that the category of orientals is the closure of the category of simplexes under certain filler operations which represent complicial operations.

1. Introduction

The orientals or oriented simplexes are a family of strict ω -categories $\mathcal{O}_0, \mathcal{O}_1, \dots$ constructed by Street [3]. In [2] it is shown that there is a full subcategory of the category of strict ω -categories which is isomorphic to a category of chain complexes with additional structure, and it is shown that the orientals lie in this full subcategory. Implicitly, this describes the morphisms between orientals. The object of the present paper is to make the description of the morphisms more explicit. It turns out that they have a very simple combinatorial description; see Theorem 4.2 below. This supports Street's assertion in [3] that the orientals are fundamental structures of nature.

Let \mathcal{O} be the category of orientals, let Δ be the simplex category, and let $\mathbf{Z}\Delta$ be the category whose morphisms are integer linear combinations of morphisms in Δ ; the morphisms in $\mathbf{Z}\Delta$ are to be composed bilinearly. Theorem 4.2 shows that \mathcal{O} is a subcategory of $\mathbf{Z}\Delta$ containing Δ . The morphisms in \mathcal{O} can therefore be expressed in terms of those in Δ by using the operations of addition and subtraction, but these operations are defined only in the larger category $\mathbf{Z}\Delta$. We will however show that \mathcal{O} is the closure of Δ under a family of operations internal to \mathcal{O} . These operations are called filler operations and they are the universal examples for Verity's complicial operations [5]. Verity shows that ω -category nerve structures on simplicial sets are equivalent to families of complicial operations with suitable properties, and our closure result was inspired by Verity's theorem.

In Section 2 we recall the relevant results relating chain complexes and ω -categories; in Section 3 we show how the individual orientals can be expressed in

terms of chain complexes; and in Section 4 we describe the morphisms between the orientals. Finally, Section 5 contains the material on fillers.

2. Chain complexes and omega-categories

In [2] there is a description of a functor ν from a category of chain complexes with additional structure to the category of strict ω -categories; it is modelled on Street's theory of parity complexes [4]. In this section we recall some results from [2] concerning the functor ν . All ω -categories in this paper are strict ω -categories.

We restrict attention to augmented chain complexes concentrated in nonnegative dimensions whose chain groups are free abelian groups with prescribed bases. Throughout this section, let $K = (K, \partial, \epsilon)$ be such a chain complex. We will say that K itself has a basis, consisting of the disjoint union of the bases for the individual chain groups. We make each chain group K_q into a partially ordered abelian group as follows: if $x \in K_q$, then $x \geq 0$ if and only if x is a sum of basis elements. We will now define the associated ω -category νK , using Street's one-sorted notion of ω -category [3, 1]; thus νK is a single set which serves as the set of morphisms for every category in a family indexed by the nonnegative integers.

DEFINITION 2.1. The members of the ω -category νK are the double sequences

$$x = (x_0^-, x_0^+ \mid x_1^-, x_1^+ \mid \dots)$$

satisfying the following conditions:

- (1) $x_q^-, x_q^+ \in K_q$;
- (2) there exists N such that $x_q^- = x_q^+ = 0$ for $q > N$;
- (3) $x_q^+ - x_q^- = \partial x_{q+1}^- = \partial x_{q+1}^+$ for $q \geq 0$;
- (4) $x_q^- \geq 0$ and $x_q^+ \geq 0$ for $q \geq 0$;
- (5) $\epsilon x_0^- = \epsilon x_0^+ = 1$.

For $p \geq 0$ we make νK into the morphism set of a category as follows: the left identity $d_p^- x$ and the right identity $d_p^+ x$ of the element $x = (x_0^-, x_0^+ \mid x_1^-, x_1^+ \mid \dots)$ are given by

$$d_p^- x = (x_0^-, x_0^+ \mid \dots \mid x_{p-1}^-, x_{p-1}^+ \mid x_p^-, x_p^- \mid 0, 0 \mid \dots)$$

and

$$d_p^+ x = (x_0^-, x_0^+ \mid \dots \mid x_{p-1}^-, x_{p-1}^+ \mid x_p^+, x_p^+ \mid 0, 0 \mid \dots);$$

if x and y are elements such that $d_p^+ x = d_p^- y$, say $d_p^+ x = d_p^- y = w$, then the composite $x \#_p y$ is given by

$$x \#_p y = x - w + y,$$

where the addition and subtraction are performed termwise.

In the multi-sorted description of νK the p -cells are the elements of the form

$$(x_0^-, x_0^+, \mid \dots \mid x_{p-1}^-, x_{p-1}^+ \mid x_p, x_p \mid 0, 0 \mid \dots).$$

REMARK 2.2. Definition 2.1 is based on the well-known equivalence between chain complexes concentrated in nonnegative dimensions and ω -category objects in the category of abelian groups; see [1] for example. Indeed, if K is a chain complex concentrated in nonnegative degrees, then the equivalent ω -category object is defined in the same way as νK , except that conditions (4) and (5) are omitted.

In order to construct some members of νK , we now introduce some notation. Given a chain x , we write ∂^+x and ∂^-x for the positive and negative parts of the boundary ∂x ; thus ∂^+x and ∂^-x are sums of basis elements with no common terms such that

$$\partial x = \partial^+x - \partial^-x.$$

Given a p -dimensional chain x , we write $\langle x \rangle$ for the double sequence given by

$$\langle x \rangle = ((\partial^-)^p x, (\partial^+)^p x \mid \dots \mid \partial^-x, \partial^+x \mid x, x \mid 0, 0 \mid \dots),$$

and we observe that $\langle x \rangle$ is a member of νK if and only if $\epsilon(\partial^-)^p x = \epsilon(\partial^+)^p x = 1$. This observation motivates the following definition; recall that the prescribed basis for K means the disjoint union of the prescribed basis for the individual chain groups.

DEFINITION 2.3. The prescribed basis for the chain complex K is *unital* if $\epsilon(\partial^-)^p b = \epsilon(\partial^+)^p b = 1$ whenever b is a basis element of dimension p . If K has a unital basis, then the elements $\langle b \rangle$ of νK corresponding to the basis elements b are called *atoms*.

The main result says that ν restricts to an equivalence if one imposes suitable restrictions on the bases of the chain complexes involved. It suffices that the bases should be both unital and strongly loop-free, where strong loop-freeness is defined as follows.

DEFINITION 2.4. The prescribed basis for the chain complex K is *strongly loop-free* if it has a partial ordering such that $a < b$ whenever a is an element with a negative coefficient in ∂b or b is an element with a positive coefficient in ∂a .

Note that the partial ordering in Definition 2.4 is quite different from the partial orderings of the individual chain groups.

By combining these restrictions we obtain a category \mathbf{F} of chain complexes as follows.

DEFINITION 2.5. The objects of \mathbf{F} are the augmented chain complexes of abelian groups concentrated in nonnegative degrees together with prescribed strongly loop-free unital bases. The morphisms of \mathbf{F} are the augmentation and order-preserving chain maps.

The main result [2, 5.11] is now as follows.

THEOREM 2.6. *The construction ν is a full and faithful embedding of the category \mathbf{F} in the category of strict ω -categories.*

We will also use the following result [2, 5.6].

THEOREM 2.7. *If K is an object of \mathbf{F} , then the ω -category νK is generated under composition by its atoms.*

3. The construction of the orientals

Let $K[n]$ be the chain complex of the standard n -simplex, where n is a nonnegative integer. We will show that $K[n]$ is an object in the category \mathbf{F} and that $\nu K[n]$ is isomorphic to the n th oriental \mathcal{O}_n . Essentially, we are working with free abelian groups whose bases are certain sets constructed by Street in [3]. The abelian group

context makes the theory rather easier, but the proof that $K[n]$ is in \mathbf{F} uses the same computations as in [3].

We recall that $K[n]$ is an augmented chain complex of free abelian groups; the prescribed basis of $K[n]_q$ consists of the ordered $(q+1)$ -tuples of integers $[a_0, \dots, a_q]$ such that $0 \leq a_0 < a_1 < \dots < a_q \leq n$; the boundary $\partial: K[n]_q \rightarrow K[n]_{q-1}$ for $q > 0$ is given by the alternating sum formula

$$\partial[a_0, \dots, a_q] = [a_1, \dots, a_q] - [a_0, a_2, \dots, a_q] + \dots + (-1)^q [a_0, \dots, a_{q-1}];$$

the augmentation is given by

$$\epsilon[a_0] = 1.$$

THEOREM 3.1. *The prescribed basis for $K[n]$ is strongly loop-free and unital, so that $K[n]$ is an object of \mathbf{F} .*

PROOF. To show that the basis is strongly loop-free, we must find a partial ordering of the basis elements such that $a < b$ if a is a term in ∂b with a negative coefficient or if b is a term in ∂a with a positive coefficient. There is in fact a total ordering with these properties; it can be described recursively as follows. Let $a = [a_0, \dots, a_p]$ and $b = [b_0, \dots, b_q]$ be distinct basis elements. Then $a < b$ if $a_0 < b_0$, or if $a_0 = b_0$ and $p = 0$, or if $a_0 = b_0$ and $p, q > 0$ and $[a_1, \dots, a_p] > [b_1, \dots, b_q]$.

To show that the basis is unital, let $a = [a_0, \dots, a_p]$ be a basis element; we must show that $\epsilon(\partial^-)^p a = \epsilon(\partial^+)^p a = 1$. In order to do this, we compute $(\partial^-)^q a$ and $(\partial^+)^q a$ for $0 < q \leq p$ inductively. We find that $(\partial^-)^q a$ and $(\partial^+)^q a$ are the sums of terms $[a_{i(0)}, \dots, a_{i(p-q)}]$ such that the indices of the omitted elements $a_{j(1)}, \dots, a_{j(q)}$ form an increasing sequence $j(1) < \dots < j(q)$ of integers of alternating parity; to be precise, we get the terms of $(\partial^-)^q a$ by taking $j(1)$ to be odd, $j(2)$ to be even, etc., and we get the terms of $(\partial^+)^q a$ by taking $j(1)$ to be even, $j(2)$ to be odd, etc. In particular, if $p > 0$ then we get $(\partial^-)^p a = [a_0]$ and $(\partial^+)^p a = [a_p]$, and these formulae are obviously valid for $p = 0$ as well. It follows that $\epsilon(\partial^-)^p a = \epsilon(\partial^+)^p a = 1$ as required. \square

In [2, 6] an axiomatic characterisation of $\nu K[n]$ is used to show that it is isomorphic to the n th oriental. Here we sketch a direct comparison.

THEOREM 3.2. *The ω -category $\nu K[n]$ is isomorphic to the n th oriental \mathcal{O}_n .*

PROOF. The elements of \mathcal{O}_n (see [3, 2]) are double sequences

$$a = (a_0^1, a_0^0 \mid a_1^1, a_1^0 \mid \dots),$$

where a_q^1 and a_q^0 are certain finite sets of q -dimensional basis elements for $K[n]$. Associated to such a double sequence a there is a double sequence

$$x = (x_0^-, x_0^+ \mid x_1^-, x_1^+ \mid \dots)$$

such that x_q^- and x_q^+ are q -dimensional chains in $K[n]$; one takes x_q^- and x_q^+ to be the sums of the members of a_q^1 and of a_q^0 respectively. From the definition of \mathcal{O}_n , one can check that x is in $\nu K[n]$, so we have obtained a function $i: \mathcal{O}_n \rightarrow \nu K[n]$, and it is clear that i is injective. By comparing the definitions, one sees that i is a morphism of ω -categories. One can also check that the atoms are in the image of i . Since $\nu K[n]$ is generated by its atoms (see Theorem 2.7), it follows that i is an isomorphism. Therefore $\nu K[n] \cong \mathcal{O}_n$ as required. \square

4. Morphisms between orientals

By combining Theorems 3.2 and 2.6, we see that the category of orientals is isomorphic to the category of augmentation and order-preserving chain maps between the chain complexes $K[n]$. In this section we get a simple description of the chain maps, and from this we get a simple description of the morphisms between the orientals.

We begin with the simplex category Δ . The objects of Δ are the nonnegative integers $0, 1, \dots$, the morphism set $\Delta(m, n)$ is the set of non-decreasing functions

$$f: \{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, n\},$$

and the morphisms are composed in the obvious way. For the morphism in $\Delta(m, n)$ given by $i \mapsto f_i$ we will use the notation (f_0, \dots, f_m) ; thus the morphisms in $\Delta(m, n)$ are the ordered $(m+1)$ -tuples of integers (f_0, \dots, f_m) such that

$$0 \leq f_0 \leq f_1 \leq \dots \leq f_m \leq n.$$

There is a standard functor from Δ to chain complexes sending n to $K[n]$, as follows. Let $f = (f_0, \dots, f_m)$ be a morphism in $\Delta(m, n)$ and let $[a(0), \dots, a(q)]$ be a basis element for $K[m]$. If $f_{a(0)} < f_{a(1)} < \dots < f_{a(q)}$ then

$$f[a(0), \dots, a(q)] = [f_{a(0)}, \dots, f_{a(q)}];$$

if $f_{a(0)}, \dots, f_{a(q)}$ are not distinct then

$$f[a(0), \dots, a(q)] = 0.$$

Now let $\mathbf{Z}\Delta$ be the category with objects $0, 1, 2, \dots$ such that the set of morphisms $\mathbf{Z}\Delta(m, n)$ is the free abelian group with basis $\Delta(m, n)$. The morphisms in $\mathbf{Z}\Delta$ are composed bilinearly; that is,

$$\left(\sum_g \mu_g g\right) \circ \left(\sum_f \lambda_f f\right) = \sum_h \left(\sum_{g \circ f = h} \mu_g \lambda_f\right) h.$$

Since the chain maps from $K[m]$ to $K[n]$ form an abelian group, the functor $n \mapsto K[n]$ from Δ to chain complexes extends to a functor on $\mathbf{Z}\Delta$; in other words, there is a natural homomorphism from $\mathbf{Z}\Delta(m, n)$ to the group of chain maps from $K[m]$ to $K[n]$. We will now show that this homomorphism is an isomorphism.

THEOREM 4.1. *The group of chain maps from $K[m]$ to $K[n]$ is naturally isomorphic to $\mathbf{Z}\Delta(m, n)$.*

PROOF. Let A_q and B_q be the bases for $K[m]_q$ and $K[n]_q$ respectively, let A'_q be the subset of A_q consisting of the elements $[a_0, \dots, a_q]$ such that $a_0 = 0$, and let $K[m]'_q$ be the subgroup of $K[m]_q$ generated by the members of A'_q . If $[a_0, \dots, a_q]$ is a basis element of $K[m]_q$ such that $a_0 > 0$, then

$$[a_0, \dots, a_q] = \partial[0, a_0, \dots, a_q] + s$$

with $s \in K[m]'_q$, so a chain map from $K[m]$ to $K[n]$ is uniquely determined by its restrictions to the subgroups $K[m]'_q$. It is therefore sufficient to show that these restrictions induce an isomorphism

$$\mathbf{Z}\Delta(m, n) \rightarrow \bigoplus_q \text{hom}(K[m]'_q, K[n]_q).$$

To do this, let f be a morphism in $\Delta(m, n)$ and let the size of the image of f be $q+1$. To f we associate basis elements $a = [a_0, \dots, a_q]$ in A'_q and $b = [b_0, \dots, b_q]$

in B_q as follows: b_0, \dots, b_q are the members of the image of f in ascending order, and a_i is the smallest member of the inverse image $f^{-1}(b_i)$. We find that these assignments produce a bijection

$$\Delta(m, n) \rightarrow \prod_q (A'_q \times B_q);$$

we also find that $f[a_0, \dots, a_q] = [b_0, \dots, b_q]$, that $f[i_0, \dots, i_r] = 0$ if $[i_0, \dots, i_r]$ is a member of A'_r for some $r > q$, and that $f[i_0, \dots, i_q] = 0$ if $[i_0, \dots, i_q]$ is a member of A' which precedes $[a_0, \dots, a_q]$ lexicographically (i.e. there exists r such that $i_0 = a_0, \dots, i_{r-1} = a_{r-1}, i_r < a_r$). It follows that the morphism

$$\mathbf{Z}\Delta(m, n) \rightarrow \bigoplus_q \text{hom}(K[m]'_q, K[n]_q)$$

is an isomorphism, as required. This completes the proof. \square

Theorem 4.1 shows that there is a category $\mathbf{Z}\Delta$ with objects $0, 1, 2, \dots$ and morphism sets $\mathbf{Z}\Delta(m, n)$. According to Theorem 2.6, the category of orientals is isomorphic to the subcategory of $\mathbf{Z}\Delta$ consisting of the augmentation and order-preserving morphisms. It is easy to characterise these morphisms, and the result is as follows.

THEOREM 4.2. *The set of morphisms from the oriental \mathcal{O}_m to the oriental \mathcal{O}_n is naturally isomorphic to the subset of $\mathbf{Z}\Delta(m, n)$ consisting of the linear combinations x such that*

- (1) *the sum of the coefficients in x is 1;*
- (2) *for all injective morphisms f in Δ with codomain m the coefficients of the injective morphisms in the composite $x \circ f$ are nonnegative.*

The subset of $\mathbf{Z}\Delta(m, n)$ described in Theorem 4.2 will be denoted $\mathcal{O}(m, n)$, and the category of orientals will be denoted \mathcal{O} ; thus \mathcal{O} has objects \mathcal{O}_n and morphism sets $\mathcal{O}(m, n)$.

EXAMPLE 4.3. If $q > 0$ and $0 \leq i_0 < i_1 < \dots < i_q \leq n$ then there is a morphism x in $\mathcal{O}(1, n)$ given by

$$x = (i_0, i_1) - (i_1, i_1) + (i_1, i_2) - (i_2, i_2) + \dots + (i_{q-1}, i_q);$$

indeed the sum of the coefficients is 1, there are no injective morphisms with negative coefficients in $x \circ (0, 1) = x$, and there are no injective morphisms with negative coefficients in $x \circ (0) = (i_0)$ or $x \circ (1) = (i_q)$ either.

Similarly there is a morphism y in $\mathcal{O}(2, 2)$ given by

$$y = (0, 1, 1) - (1, 1, 1) + (1, 1, 2).$$

5. Fillers

According to Theorem 4.2, the morphisms in the category \mathcal{O} of orientals are integer linear combinations of morphisms in the subcategory Δ ; but one cannot define addition and subtraction inside \mathcal{O} , only in the larger category $\mathbf{Z}\Delta$. In this section we will show how to generate \mathcal{O} from Δ by using operations defined inside \mathcal{O} .

We first observe that the sets $\mathbf{Z}\Delta(m, n)$ for a fixed n and varying m form a simplicial set, and that the $\mathcal{O}(m, n)$ form a simplicial subset. That is to say, there

are face operations $\partial_i: \mathbf{Z}\Delta(m, n) \rightarrow \mathbf{Z}\Delta(m-1, n)$ for $m > 0$ and $0 \leq i \leq m$ given by

$$\partial_i x = x \circ (0, \dots, i-1, i+1, \dots, m),$$

there are degeneracy operations $\epsilon_i: \mathbf{Z}\Delta(m, n) \rightarrow \mathbf{Z}\Delta(m+1, n)$ for $0 \leq i \leq m$ given by

$$\epsilon_i x = x \circ (0, \dots, i-1, i, i, i+1, \dots, m),$$

the subcategory \mathcal{O} is closed under these operations, and the usual simplicial identities apply. In this paper we will use the following simplicial identities:

$$\partial_i \epsilon_{i+1} = \epsilon_i \partial_i, \quad \partial_i \epsilon_i = \partial_{i+1} \epsilon_i = \text{id}, \quad \partial_{i+2} \epsilon_i = \epsilon_i \partial_{i+1}.$$

We now define two further families of operations.

DEFINITION 5.1. Let x and y be morphisms in $\mathbf{Z}\Delta(m, n)$ such that $\partial_i x = \partial_{i+1} y$ for some i with $0 \leq i \leq m-1$. Then $x \nabla_i y$ is the morphism in $\mathbf{Z}\Delta(m+1, n)$ given by

$$x \nabla_i y = \epsilon_{i+1} x - \epsilon_i \partial_i x + \epsilon_i y$$

and $x \vee_i y$ is the morphism in $\mathbf{Z}\Delta(m, n)$ given by

$$x \vee_i y = x - \epsilon_i \partial_i x + y.$$

We call $x \nabla_i y$ a *filler* and $x \vee_i y$ a *pasting*.

For example, let x and y be the morphisms of Example 4.3; then

$$x = (i_0, i_1) \vee_0 \dots \vee_0 (i_{q-1}, i_q)$$

and $y = (0, 1) \nabla_0 (1, 2)$. (No brackets are needed in the expression for x , since one finds that \vee_0 is associative.) As models for use in later arguments, we also give the following examples.

EXAMPLE 5.2. If $a \leq b$ then

$$\begin{aligned} (a, a) \nabla_0 (a, b) &= (a, a, b), \\ (a, b) \nabla_0 (b, b) &= (a, b, b), \\ (a, a) \vee_0 (a, b) &= (a, b), \\ (a, b) \vee_0 (b, b) &= (a, b). \end{aligned}$$

From the simplicial identities we deduce the following result.

PROPOSITION 5.3. If x and y are morphisms in $\mathbf{Z}\Delta(m, n)$ such that $\partial_i x = \partial_{i+1} y$, then

$$\begin{aligned} \partial_i (x \nabla_i y) &= y, \\ \partial_{i+1} (x \nabla_i y) &= x \vee_i y, \\ \partial_{i+2} (x \nabla_i y) &= x. \end{aligned}$$

The object of this section is to show that the category \mathcal{O} is the closure of its subcategory Δ under the operations ∇_i . We begin by verifying that \mathcal{O} is closed under the filler and pasting operations.

PROPOSITION 5.4. If x and y are morphisms in $\mathcal{O}(m, n)$ such that $\partial_i x = \partial_{i+1} y$ then $x \nabla_i y \in \mathcal{O}(m+1, n)$ and $x \vee_i y \in \mathcal{O}(m, n)$.

PROOF. Since $x \vee_i y = \partial_{i+1}(x \nabla_i y)$, it suffices to consider $x \nabla_i y$. It is clear that the sum of the coefficients in $x \nabla_i y$ is equal to 1, and it remains to show that injective morphisms have nonnegative coefficients in $(x \nabla_i y) \circ f$ when f is an injective morphism in Δ with codomain $m+1$. If $i+2$ is not in the image of f , then this holds because $\partial_{i+2}(x \nabla_i y) = x \in \mathcal{O}(m, n)$. If i is not in the image of f , then this holds similarly because $\partial_i(x \nabla_i y) = y$. If both i and $i+2$ occur in the image of f but $i+1$ does not, then the injective terms in $(x \nabla_i y) \circ f$ are obtained by adding those in $(\epsilon_{i+1}x) \circ f$ and $(\epsilon_i y) \circ f$, so again they have nonnegative coefficients. Finally, if $i, i+1$ and $i+2$ are all in the image of f , then $(x \nabla_i y) \circ f$ has no injective terms at all. This completes the proof. \square

Conversely, we now give a sequence of results designed to express the morphisms of \mathcal{O} in terms of fillers and pastings.

PROPOSITION 5.5. *Let x be a morphism in $\mathbf{Z}\Delta(m, n)$ with $m > 0$ such that injective morphisms have nonnegative coefficients in $x \circ f$ for all injective morphisms f in Δ with codomain m . In the obvious notation, write*

$$x = (x_0, 0) + \dots + (x_n, n)$$

with $x_i \in \mathbf{Z}\Delta(m-1, i)$. Then injective morphisms have nonnegative coefficients in

$$(x_r + x_{r+1} + \dots + x_n) \circ g$$

for $0 \leq r \leq n$ and for all injective morphisms g in Δ with codomain $m-1$.

PROOF. Let $g = (g_0, \dots, g_q)$, and let $a = (a_0, \dots, a_q)$ be an injective morphism in $\Delta(q, n)$. If $a_q < r$ then the coefficient of a in $(x_r + \dots + x_n) \circ g$ is the sum over $r \leq i \leq n$ of the coefficients of (a_0, \dots, a_q, i) in $x \circ (g_0, \dots, g_q, m)$; if $a_q \geq r$ then the coefficient of a in $(x_r + \dots + x_n) \circ g$ is the same as its coefficient in $x \circ (g_0, \dots, g_q)$. The result follows. \square

PROPOSITION 5.6. *Let x be a morphism in $\mathbf{Z}\Delta(m, n)$ such that injective morphisms have nonnegative coefficients in $x \circ f$ for all injective morphisms f in Δ with codomain m and such that the sum of the coefficients in x is zero. Then $x = 0$.*

PROOF. We use induction on m . Note first that every morphism in $\Delta(0, n)$ is injective, so $x \circ (m)$ is a linear combination with nonnegative coefficients such that the sum of the coefficients is zero. It follows that $x \circ (m) = 0$.

Suppose that $m = 0$. Then $x = x \circ (m)$, so $x = 0$.

Now suppose that $m > 0$. Write x in the form

$$x = (x_0, 0) + \dots + (x_n, n)$$

as in Proposition 5.5. Since $x \circ (m) = 0$, the sum of the coefficients in x_i is zero for $0 \leq i \leq m$. From Proposition 5.5 and the inductive hypothesis, it follows that $x_r + x_{r+1} + \dots + x_n = 0$ for $0 \leq r \leq n$, and it then follows that $x_i = 0$ for $0 \leq i \leq n$. Therefore $x = 0$, as required.

This completes the proof. \square

PROPOSITION 5.7. *Let x be a morphism in $\mathcal{O}(m, n)$, let s be the smallest integer to appear in the terms of x , and let t be the largest integer to appear in the terms of x . Then $x[0] = [s]$ and $x[m] = [t]$.*

PROOF. Since every morphism in $\Delta(0, n)$ is injective, $x \circ (0)$ and $x \circ (m)$ are linear combinations with nonnegative integer coefficients. Since the sums of the coefficients are 1, they reduce to single terms, so that $x \circ (0) = (a)$ and $x \circ (m) = (b)$ for some integers a and b . It follows that $x[0] = [a]$ and $x[m] = [b]$, and it now suffices to show that $s = a$ and $t = b$. We will show that $t = b$; the proof that $s = a$ is similar.

Suppose first that $m = 0$. Then $x = x \circ (m) = (b)$, so $t = b$.

Now suppose that $m > 0$. Write x in the form

$$x = (x_0, 0) + \dots + (x_n, n)$$

as in Proposition 5.5. Since t is the largest integer to appear in the terms of x , we have $x_t \neq 0$ and $x_i = 0$ for $t < i \leq n$. It follows from Proposition 5.5 that the injective terms in $x_t \circ f$ have nonnegative coefficients for all injective morphisms f in Δ with codomain $m - 1$. Since $x_t \neq 0$, it follows from Proposition 5.6 that the sum of the coefficients of x_t is not zero; in other words $[t]$ has a non-zero coefficient in $x[m]$. Since $x[m] = [b]$, we must have $t = b$ as required. This completes the proof. \square

PROPOSITION 5.8. *Let x be a morphism in $\mathcal{O}(m, n)$ with $x[m] = [t]$ such that (t, \dots, t) has a non-zero coefficient in x . Then $x = (t, \dots, t)$.*

PROOF. By Proposition 5.7, t is the largest integer to appear in the terms of x . The coefficient of $[t]$ in $x[0]$ is therefore the coefficient of (t, \dots, t) in x , which we are assuming to be non-zero. By Proposition 5.7 again, t is also the smallest integer to appear in the terms of x . It follows that $x = (t, \dots, t)$, as required. \square

PROPOSITION 5.9. *Let x be a morphism in $\mathcal{O}(m, n)$ with $x[m] = [t]$, and let r be an integer with $0 \leq r \leq m - 2$ such that x is a linear combination of terms (a_0, \dots, a_m) with $a_r < t$. Then there is a factorisation*

$$x = u \vee_r v = u \vee_r (\partial_{r+2} v \nabla_r \partial_r v)$$

in \mathcal{O} such that $u[m] = [t]$ and u is a linear combination of terms (a_0, \dots, a_m) with $a_{r+1} < t$.

PROOF. Let $\alpha, \beta: \mathbf{Z}\Delta(m, n) \rightarrow \mathbf{Z}\Delta(m, n)$ be the linear functions given on morphisms (a_0, \dots, a_m) in $\Delta(m, n)$ as follows:

$$\alpha(a_0, \dots, a_m) = \begin{cases} (a_0, \dots, a_{r-1}, a_r, a_{r+1}, a_{r+2}, \dots, a_m) & \text{if } a_{r+1} < t, \\ (a_0, \dots, a_{r-1}, a_r, a_r, a_{r+2}, \dots, a_m) & \text{if } a_{r+1} \geq t, \end{cases}$$

$$\beta(a_0, \dots, a_m) = \begin{cases} (a_0, \dots, a_{r-1}, a_{r+1}, a_{r+1}, a_{r+2}, \dots, a_m) & \text{if } a_{r+1} < t, \\ (a_0, \dots, a_{r-1}, a_r, a_{r+1}, a_{r+2}, \dots, a_m) & \text{if } a_{r+1} \geq t. \end{cases}$$

For a morphism $a = (a_0, \dots, a_m)$ in $\Delta(m, n)$ with $a_r < t$ and $a_m \leq t$ we find that

$$a = \alpha a \vee_r \beta a = \alpha a \vee_r (\partial_{r+2} \beta a \nabla_r \partial_r \beta a)$$

(the calculation is as in Example 5.2). Since x is a linear combination of morphisms of this type, we get a factorisation

$$x = u \vee_r v = u \vee_r (\partial_{r+2} v \nabla_r \partial_r v)$$

in $\mathbf{Z}\Delta(m, n)$, where $u = \alpha x$ and $v = \beta x$. Clearly $u[m] = x[m]$, so that $u[m] = [t]$, and it remains to show that the factors are in \mathcal{O} . Since \mathcal{O} is closed under the operations ∂_i , it suffices to show that u and v are in $\mathcal{O}(m, n)$ by verifying the

conditions of Theorem 4.2. Now, it is clear that the sums of the coefficients in u and v are equal to 1, so it remains to consider $u \circ f$ and $v \circ f$ when $f = (f_0, \dots, f_q)$ is injective with codomain m ; we must show that every injective morphism $i = (i_0, \dots, i_q)$ with codomain n has nonnegative coefficients in $u \circ f$ and $v \circ f$.

If $r+1$ is not in the image of f , then $u \circ f = x \circ f$, so the result holds for $u \circ f$. Similarly, if r is not in the image of f , then $v \circ f = x \circ f$, so the result holds for $v \circ f$.

Now suppose that both r and $r+1$ are in the image of f , say $f_{j-1} = r$ and $f_j = r+1$. If $i_j < t$, then the coefficient of i in $u \circ f$ is as in $x \circ f$ and the coefficient of i in $v \circ f$ is zero. If $i_j \geq t$ then the coefficient of i in $u \circ f$ is zero and the coefficient of i in $v \circ f$ is as in $x \circ f$. In both cases, the coefficients of i in $u \circ f$ and $v \circ f$ are therefore nonnegative.

Next we consider $u \circ f$ when the image of f contains $r+1$ but not r , say $f_j = r+1$. If $i_j \geq t$, then the coefficient of i in $u \circ f$ is zero; if $i_j < t$ and $i_k < t$ for some $k > j$, then the coefficient of i in $u \circ f$ is the same as in $x \circ f$; if $j = q-1$ and $i_j < t$ and $i_q = t$, then the coefficient of i in $u \circ f$ is the sum of the coefficients of i in $x \circ f$ and in $x \circ (f_0, \dots, f_{q-2}, r, r+1)$; if $j = q$ and $i_j < t$, then the coefficient of i in $u \circ f$ is the sum of the coefficients of i in $x \circ f$ and of (i_0, \dots, i_q, t) in $x \circ (f_0, \dots, f_{q-1}, r, r+1)$. In all cases the coefficient of i in $u \circ f$ is nonnegative.

The remaining case, $v \circ f$ when the image of f contains r but not $r+1$, is equivalent to the previous case, because $v \circ f = u \circ f'$, where f' is got from f by changing r to $r+1$.

This completes the proof. \square

PROPOSITION 5.10. *Let x be a morphism in $\mathcal{O}(m, n)$ with $m > 0$ and $x[m] = [t]$ such that x is a linear combination of terms (a_0, \dots, a_m) with $a_{m-1} < t$. Then there is a factorisation*

$$x = u \vee_{m-1} v$$

in \mathcal{O} such that $u[m] = [t']$ with $t' < t$ and $v[m] = [t]$ and v is a linear combination of terms (a_0, \dots, a_m) with $a_{m-1} = a_m$ or $a_m = t$.

PROOF. We get a factorisation $x = u \vee_{m-1} v$ in \mathcal{O} just as in the proof of Proposition 5.9, except that r is replaced by $m-1$, and we find that u and v are as described. \square

PROPOSITION 5.11. *Let x be a morphism in $\mathcal{O}(m, n)$ with $x[m] = [t]$, and let r be an integer with $0 \leq r \leq m-2$ such that x is a linear combination of terms (a_0, \dots, a_m) with $a_{r+1} = a_m$ or $a_m = t$. Then there is a factorisation*

$$x = u \vee_r v = (\partial_{r+2} u \nabla_r \partial_r u) \vee_r v$$

in \mathcal{O} such that $v[m] = [t]$ and v is a linear combination of terms (a_0, \dots, a_m) with $a_r = a_m$ or $a_m = t$.

PROOF. Let $\alpha, \beta: \mathbf{Z}\Delta(m, n) \rightarrow \mathbf{Z}\Delta(m, n)$ be the linear functions given on morphisms (a_0, \dots, a_m) in $\Delta(m, n)$ as follows:

$$\begin{aligned} \alpha(a_0, \dots, a_m) &= \begin{cases} (a_0, \dots, a_{r-1}, a_r, a_{r+1}, a_{r+2}, \dots, a_m) & \text{if } a_m < t, \\ (a_0, \dots, a_{r-1}, a_r, a_r, a_{r+2}, \dots, a_m) & \text{if } a_m \geq t, \end{cases} \\ \beta(a_0, \dots, a_m) &= \begin{cases} (a_0, \dots, a_{r-1}, a_{r+1}, a_{r+1}, a_{r+2}, \dots, a_m) & \text{if } a_m < t, \\ (a_0, \dots, a_{r-1}, a_r, a_{r+1}, a_{r+2}, \dots, a_m) & \text{if } a_m \geq t \end{cases} \end{aligned}$$

(this is the same as in the proof of Proposition 5.9, except that the splitting depends on a_m rather than a_{r+1}). We now get a factorisation

$$x = u \vee_r v = (\partial_{r+2} u \nabla_r \partial_r u) \vee_r v$$

in $\mathbf{Z}\Delta(m, n)$, where $u = \alpha x$ and $v = \beta x$. It is clear that $v[m] = x[m]$, so that $v[m] = [t]$, and it is clear that v is a linear combination of terms (a_0, \dots, a_m) with $a_r = a_m$ or $a_m = t$. It therefore remains to show that u and v are in $\mathcal{O}(m, n)$. Obviously the sums of the coefficients in u and v are equal to 1, so it suffices to show that an injective morphism $i = (i_0, \dots, i_q)$ with codomain n has a nonnegative coefficient in $u \circ f$ and $v \circ f$ whenever $f = (f_0, \dots, f_q)$ is an injective morphism with codomain m . The case $u \circ f$ with $r+1$ not in the image of f and the case $v \circ f$ with r not in the image of f are handled as in the proof of Proposition 5.9. Also as in the proof of that proposition, the case $u \circ f$ with the image of f containing $r+1$ but not r reduces to the case $v \circ f$ with the image of f containing r but not $r+1$. In the remaining cases, we argue as follows.

Suppose that the image of f contains r and $r+1$. If $f_q > r+1$, then the coefficient of i in $u \circ f$ is zero and the coefficient of i in $v \circ f$ is as in $x \circ f$. If $f_{q-1} = r$ and $f_q = r+1$ and $i_q < t$ then the coefficient of i in $u \circ f$ is the coefficient of i in $x \circ (f_0, \dots, f_{q-1}, m)$ and the coefficient of i in $v \circ f$ is the coefficient of (i_0, \dots, i_q, t) in $x \circ (f_0, \dots, f_q, m)$. If $f_{q-1} = r$ and $f_q = r+1$ and $i_q \geq t$ then the coefficient of i in $u \circ f$ is zero and the coefficient of i in $v \circ f$ is as in $x \circ f$.

Now suppose that the image of f contains r but not $r+1$, say $f_j = r$. If $i_q \geq t$, then the coefficient of i in $v \circ f$ is the same as in $x \circ f$. If $i_q < t$ and $j < q$ and $f_q = m$, then the coefficient of i in $v \circ f$ is zero. If $i_q < t$ and $j < q$ and $f_q < m$, then the coefficient of i in $v \circ f$ is the coefficient of (i_0, \dots, i_q, t) in $x \circ (f_0, \dots, f_q, m)$. Finally, if $i_q < t$ and $j = q$, then the coefficient of i in $v \circ f$ is the coefficient of i in $x \circ (f_0, \dots, f_{q-1}, m)$ plus the coefficient of (i_0, \dots, i_q, t) in $x \circ (f_0, \dots, f_q, m)$. This completes the proof. \square

PROPOSITION 5.12. *Let x be a morphism in $\mathcal{O}(m, n)$ with $x[m] = [t]$ such that x is a linear combination of terms (a_0, \dots, a_m) with $a_0 = a_m$ or $a_m = t$. Then x is a linear combination of terms (a_0, \dots, a_m) with $a_m = t$.*

PROOF. For $a < t$, we must show that (a, \dots, a) has zero coefficient in x . But this holds because the coefficient of (a, \dots, a) in x is the coefficient of $[a]$ in $x[m]$, and this coefficient is zero by hypothesis. \square

We now combine these results to get factorisations for arbitrary morphisms in \mathcal{O} .

THEOREM 5.13. *Let x be a morphism in \mathcal{O} . Then x can be factorised into morphisms in Δ by using the operations ∇_i and \vee_i .*

PROOF. By Proposition 5.7, there is an integer t with $0 \leq t \leq n$ such that $x[m] = [t]$, and t is the largest integer to appear in the terms of x . We will use induction on m , and for a fixed m we will use induction on t .

Suppose first that x has a term (t, \dots, t) (this includes the base cases with $m = 0$ or $t = 0$). By Proposition 5.8, x is equal to (t, \dots, t) , so x is already a morphism in Δ .

From now on, suppose that x has no term (t, \dots, t) . It follows that x is a linear combination of terms (a_0, \dots, a_m) with $a_0 < t$. We now apply Propositions

5.9–5.11. By applying Proposition 5.9 for $r = 0, r = 1, \dots, r = m - 2$ successively, we get a factorisation

$$x = [\dots [\tilde{x} \vee_{m-2} (v'_{m-2} \nabla_{m-2} v''_{m-2})] \vee_{m-3} \dots] \vee_0 (v'_0 \nabla_0 v''_0)$$

in \mathcal{O} such that $\tilde{x}[m] = [t]$ and \tilde{x} is a linear combination of terms (a_0, \dots, a_m) with $a_{m-1} < t$. By applying Proposition 5.10 we get a factorisation

$$\tilde{x} = y \vee_{m-1} \tilde{y}$$

in \mathcal{O} such that $y[m] = [t']$ with $t' < t$ and $\tilde{y}[m] = [t]$ and \tilde{y} is a linear combination of terms (a_0, \dots, a_m) with $a_{m-1} = a_m$ or $a_m = t$. Finally, by applying Proposition 5.11 for $r = m - 2, r = m - 3, \dots, r = 0$ successively, we get a factorisation

$$\tilde{y} = (u'_{m-2} \nabla_{m-2} u''_{m-2}) \vee_{m-2} [\dots \vee_1 [(u'_0 \nabla_0 u''_0) \vee_0 z] \dots]$$

in \mathcal{O} such that $z[m] = [t]$ and z is a linear combination of terms (a_0, \dots, a_m) with $a_0 = a_m$ or $a_m = t$.

We have now got a factorisation of x in \mathcal{O} with factors $v'_r, v''_r, y, u'_r, u''_r$ and z . The factors v'_r, v''_r, u'_r and u''_r are in $\mathcal{O}(m - 1, n)$, so they can be factorised into morphisms in Δ by the induction on m . The factor y is in $\mathcal{O}(m, n)$ with $y[m] = [t']$ for some $t' < t$, so it can be factorised into morphisms in Δ by the induction on t . By Proposition 5.12, the factor z is a linear combination of terms (a_0, \dots, a_m) with $a_m = t$, so it can be expressed as (z_t, t) in the notation of Proposition 5.5. Here $z_t = \partial_m z$, so z_t is in $\mathcal{O}(m - 1, n)$. By the induction on m , we can factorise z_t into morphisms in Δ , and this clearly induces a similar factorisation for z . We have now factorised all the factors of x , so we have a factorisation of x itself. This completes the proof. \square

For example, suppose that $x \in \mathcal{O}(1, n)$. Then the factorisation reduces to repeated application of Proposition 5.10, and it takes the form

$$x = (i_0, i_0) \vee_0 (i_0, i_1) \vee_0 \dots \vee_0 (i_{q-1}, i_q),$$

where $q \geq 0$ and $0 \leq i_0 < i_1 < \dots < i_q \leq n$; in other words, $x = (i_0, i_0)$ or $x = (i_0, i_1) \vee_0 \dots \vee_0 (i_{q-1}, i_q)$ as in Example 4.3.

Finally, from Theorem 5.13, we get the main result.

THEOREM 5.14. *The category of orientals \mathcal{O} is the closure of the subcategory of simplexes Δ under the filler operations and composition.*

PROOF. By Proposition 5.4, \mathcal{O} is closed under the filler and pasting operations. Theorem 5.13 therefore shows that \mathcal{O} is the closure of Δ under the filler and pasting operations. If $x \vee_i y$ is a pasting, then $x \vee_i y = \partial_{i+1}(x \nabla_i y)$ by Proposition 5.3, so $x \vee_i y$ has the form $(x \nabla_i y) \circ \delta$ with δ a morphism in Δ . Therefore \mathcal{O} is the closure of Δ under fillers and composition. This completes the proof. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GLASGOW, UNIVERSITY GARDENS, GLASGOW,
SCOTLAND G12 8QW
E-mail address: `r.steiner@maths.gla.ac.uk`